Math 665 • FINAL EXAM • May 13, 2010

1) Categorize all zeros and singularities of the following functions, find two lowest-order non-zero terms in the Laurent or Taylor series of $f(z)$ near the given point z_o, and state the region on which the corresponding expansion is valid:

(a) $f(z)=\dfrac{\sinh z}{1-\cos z}$ at $z_0=0$

- Zeros at $z_k=i\pi k,\ k\in\mathbb{Z},\ k\neq 0$
- Poles of order 2 at $\cos z_k=1 \ \Rightarrow\ z_k=2\pi k,\ k\in\mathbb{Z},\ k\neq 0$
- Simple pole at $z=0$, with the following Laurent expansion, converging in $0<|z|<2\pi$

$$f(z)=\frac{\sinh z}{1-\cos z}=\frac{z+\frac{z^3}{3!}+O(z^5)}{1-\left(1-\frac{z^2}{2}+\frac{z^4}{4!}+O(z^6)\right)}=\frac{z+\frac{z^3}{3!}+O(z^5)}{\frac{z^2}{2}-\frac{z^4}{4!}+O(z^6)}=\frac{z\left(1+\frac{z^2}{3!}+O(z^4)\right)}{\frac{z^2}{2}\left(1-\frac{z^2}{12}+O(z^4)\right)}$$

$$=\frac{2}{z}\frac{1+\frac{z^2}{3!}+O(z^4)}{1-\underbrace{\frac{z^2}{12}+O(z^4)}_{\zeta}}=\frac{2}{z}\left(1+\frac{z^2}{3!}+O(z^4)\right)\underbrace{\left(1+\frac{z^2}{12}+O(z^4)\right)}_{1+\zeta+\zeta^2+\ldots}=\frac{2}{z}\left(1+\frac{z^2}{4}+O(z^4)\right)=\boxed{\frac{2}{z}+\frac{z}{2}+O(z^3)}$$

Residue equals 2, as it should

(b) $f(z)=\dfrac{\exp(1/z)}{\log_\pi z}$ at $z_0=1$ (branch $\log_\pi z$ satisfies $-\pi\le\arg z<\pi$)

- Branch point at $z=0$ (note: it is not an essential singularity since it is not isolated)
- Branch cut along the negative real axis
- Simple pole at $z=1$, with the following Laurent expansion, converging in $0<|z-1|<1$

Denote $z=1+\zeta$:

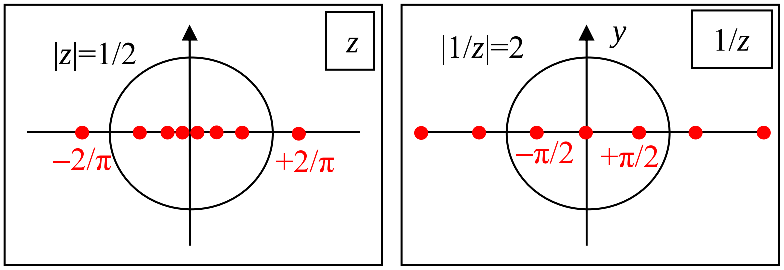
$$f(z)=\frac{\exp\frac{1}{1+\zeta}}{\log_\pi(1+\zeta)}=\frac{\exp(1-\zeta+\zeta^2+\ldots)}{\zeta-\frac{\zeta^2}{2}+\frac{\zeta^3}{3}-\frac{\zeta^4}{4}+\ldots}=\frac{\exp(1)\,\exp(-\zeta+\zeta^2+\ldots)}{\zeta\left(1-\frac{\zeta}{2}+O(\zeta^2)\right)}=\frac{e}{\zeta}\frac{1-\zeta+O(\zeta^2)}{1-\frac{\zeta}{2}+O(\zeta^2)}$$

$$=\frac{e}{\zeta}\left(1-\zeta+O(\zeta^2)\right)\left(1+\frac{\zeta}{2}+O(\zeta^2)\right)=\frac{e}{\zeta}\left(1-\frac{\zeta}{2}+O(\zeta^2)\right)=\boxed{\frac{e}{z-1}-\frac{e}{2}+O(z-1)}$$

This agrees with the value of the residue (use $N(z_0)/D'(z_0)$): $\left.\dfrac{\exp(1/z)}{(\log_{-\pi} z)'}\right|_1=\left.\dfrac{\exp(1/z)}{1/z}\right|_1=\dfrac{\exp(1/1)}{1}=e$

2) Describe all singularities of the integrand inside the integration contour, and calculate each integral. Each integration contour is a circle of radius 1/2:

(a) $\oint_{|z|=1/2} \frac{z}{\cos(1/z)} dz$



- Simple poles at $\cos(1/z)=0 \Rightarrow \frac{1}{z_k} = \pi\left(k+\frac{1}{2}\right) \Rightarrow z_k = \frac{1}{\pi(k+1/2)},\ k \in \mathbb{Z}$
- These poles have an accumulation point (a cluster point) at $z=0$

To calculate the integral, we have to use the mapping $\zeta=1/z$

$$\oint_{|z|=1/2} \frac{z\,dz}{\cos(1/z)} = -\oint_{|\zeta|=2} \frac{1}{\zeta}\frac{-d\zeta/\zeta^2}{\cos\zeta} = \oint_{|\zeta|=2} \underbrace{\frac{d\zeta}{\zeta^3\cos\zeta}}_{\text{THREE POLES INSIDE CIRCLE}} = 2\pi i\left\{\mathrm{Res}(f;0) + \mathrm{Res}\left(f;\frac{\pi}{2}\right) + \mathrm{Res}\left(f;-\frac{\pi}{2}\right)\right\}$$

Residue at zero equals 1/2, most easily calculated using series expansion:

$$\frac{1}{\zeta^3\cos\zeta} = \frac{1}{\zeta^3\left(1-\frac{\zeta^2}{2}+O(\zeta^4)\right)} = \frac{1}{\zeta^3}\left(1+\frac{\zeta^2}{2}+O(\zeta^4)\right) = \frac{1}{\zeta^3}+\frac{1}{2\zeta}+O(\zeta)$$

$$\Rightarrow 2\pi i\left\{\mathrm{Res}(f;0) + \mathrm{Res}\left(f;\frac{\pi}{2}\right) + \mathrm{Res}\left(f;-\frac{\pi}{2}\right)\right\} = 2\pi i\left(\frac{1}{2} - \left.\frac{1}{\zeta^3\sin\zeta}\right|_{\pi/2} - \left.\frac{1}{\zeta^3\sin\zeta}\right|_{-\pi/2}\right)$$

$$= 2\pi i\left(\frac{1}{2} - 2\left(\frac{2}{\pi}\right)^3\right) = \boxed{i\left(\pi - \frac{32}{\pi^2}\right)}$$

---

b) $\oint_{|z|=1/2} \frac{\cos(1/z)\,dz}{z}$

- The only singularity is the essential singularity at $z=0$. Therefore, we have to use the series expansion:

$$\frac{\cos(1/z)}{z} = \frac{1-\frac{1}{2z^2}+\frac{1}{4!z^4}+O(z^{-6})}{z} = \frac{1}{z} - \frac{1}{2z^3} + \frac{1}{4!z^5} + \ldots$$

The residue is obviously 1, so the integral over any circle surrounding the origin equals $\boldsymbol{2\pi i}$

---

3) Calculate any **two** of the following three integrals. **Carefully explain each step.**

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi(b-a)}{2}$, where $a > 0, b > 0$ are real constants (use indented contour)

$$\oint \frac{e^{iaz} - e^{ibz}}{z^2}\,dz = \int_{\varepsilon}^{R} \frac{e^{iax} - e^{ibx}}{x^2}\,dx + \int_{-R}^{\varepsilon} \frac{e^{iax} - e^{ibx}}{x^2}\,dx + \underbrace{\int_{C_\varepsilon} \frac{e^{iaz} - e^{ibz}}{z^2}\,dz}_{\substack{\to -i\pi \,\mathrm{Res}\left(\frac{e^{iaz}-e^{ibz}}{z^2};0\right) \\ = -i\pi(ia-ib) = \pi(a-b)}} + \underbrace{\int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2}\,dz}_{\substack{|\ldots| \le \frac{1}{R^2}\pi R \to 0 \\ \text{as } R\to\infty}} = 0$$

Thus, in the limit $\varepsilon \to 0$ and $R \to \infty$, we obtain (here P.V. stands for Cauchy Principal Value):

$$\text{P.V.}\int_{-\infty}^{+\infty} \frac{e^{iax} - e^{ibx}}{x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx + i\,\underbrace{\text{P.V.}\int_{-\infty}^{+\infty} \frac{\sin(ax) - \sin(bx)}{x^2}\,dx}_{=0} = \boxed{-\pi(a-b)}$$

$$\Rightarrow \boxed{\int_{0}^{+\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi(b-a)}{2}}$$

(b) $\displaystyle\int_{0}^{\infty} \frac{dx}{x^m + 1} = \frac{\pi}{m\sin(\pi/m)}$, where $m > 0$ is an integer (integrate around a circular sector)

Integrate around circular sector with angle $2\pi/m$, since along the top part of sector $z^m = (e^{i2\pi/m}x)^m = x^m$

$$\oint \frac{dz}{1+z^m} = \int_0^R \frac{dx}{1+x^m} + \int_R^0 \frac{e^{i2\pi/m}dx}{1+x^m} + \underbrace{\int_{C_R} \frac{dz}{1+z^m}}_{\substack{|\ldots| \le \frac{2\pi R/m}{R^5 - 1} \\ \to \frac{2\pi}{mR^4} \to 0}} = 2\pi i\ \mathrm{Res}\left(\frac{1}{1+z^m}; e^{-i\pi/m}\right)$$

Taking the limit $R \to \infty$:

$$(1 - e^{i2\pi/m})\int_0^\infty \frac{dx}{1+x^m} = 2\pi i \left.\frac{1}{mz^{m-1}}\right|_{e^{i\pi/m}} = \frac{2\pi i}{me^{i\pi(m-1)/m}} = \frac{2\pi i}{-me^{-i\pi/m}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^m} = \frac{2\pi i}{-me^{-i\pi/m}(1 - e^{i2\pi/m})} = \frac{2\pi i}{m(e^{i\pi/m} - e^{-i\pi/m})} = \boxed{\frac{\pi}{m\sin(\pi/m)}}$$

(c) $\displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi \ln a}{2a}$, $a > 0$ (Intergate $\log_p z/(z^2+a^2)$ around a semi-circular indented contour)

$$\oint \frac{\log_p z\,dz}{z^2+a^2} = \underbrace{\int_R^\varepsilon \frac{(\ln r + i\pi)(-dr)}{r^2+a^2}}_{\substack{z=r\,e^{i\pi} \\ dz=e^{i\pi}dr=-dr \\ \log_p z=\ln r+i\pi}} + \int_\varepsilon^R \frac{\ln x\,dx}{x^2+a^2} + \underbrace{\int_{C_\varepsilon} \frac{\log_p z\,dz}{z^2+a^2}}_{\substack{|\ldots|\le\frac{(\ln(1/\varepsilon)+\pi)}{a^2-\varepsilon^2}\pi\varepsilon \\ \sim\frac{\ln(1/\varepsilon)}{1/\varepsilon}\to 0 \text{ as } \frac{1}{\varepsilon}\to\infty}} + \underbrace{\int_{C_R} \frac{\log_p z\,dz}{z^2+a^2}}_{\substack{|\ldots|\le\frac{(\ln R+\pi)}{R^2-a^2}\pi R \\ \sim\frac{\ln R}{R}\to 0 \text{ as } R\to\infty}}$$

$$= 2\pi i \operatorname{Res}\left(\frac{\log_p z}{z^2+a^2}; ia\right) = 2\pi i \frac{\log_p(ia)}{2ia} = \frac{\pi}{a}\left(\ln a + \frac{i\pi}{2}\right)$$

Now take the limit $\varepsilon \to 0,\ R \to \infty$: $\displaystyle\int_0^\infty \frac{\ln x + (\ln x + i\pi)}{x^2+a^2}dx = 2\int_0^\infty \frac{\ln x}{x^2+a^2}dx + i\pi\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{a}\left(\ln a + \frac{i\pi}{2}\right)$

Take the real part, and divide by 2: $\boxed{\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi \ln a}{2a}}$

---

4) Some of the statements in (a)-(d) below are false. For each false statement, give a counter-example proving that it isn't true. For each true statement, state the theorem from which it follows:

(a) If the integral of $f(z)$ is zero over any closed contour in domain D, then the second derivative of $f(z)$ exists in D, even if D is *not* simply-connected

True: this follows from the Morera's Theorem, combined with the Cauchy Integral Formula.

(b) If $f(z)$ has a derivative in arbitrary domain D, it must also have an anti-derivative everywhere in D

Not true: only holds for simply-connected domains (in which case it follows from the Cauchy-Goursat theorem, and expression for anti-derivative). Consider for instance $f(z)=1/z$. It is analytic in any ring centered at the origin, but its anti-derivative has a branch cut crossing any such ring.

(c) Two contour integrals of $f(z)$ over different open contours connecting the same two points are equal if $f(z)$ is analytic along each of these two contours

Not true, since there may be a singularity with non-zero residue **between** the two contours: in this case the difference between the two integrals is a closed-contour integral with non-zero value determined by the residue(s)

(d) Integral of an analytic function $f(z)$ over a circle equals twice the integral over a semi-circle.

Not true, unless the anti-derivative is even with respect to semi-circle center, in which case both integrals equal zero. Otherwise, the closed-contour integral is zero, while the semi-circle integral is non-zero. Consider for instance $f(z)=C=const$ (or any even power of z) over a circle centered at the origin.

---

5) Find coefficients C_{-2} and C_{-4} in the Laurent series for $f(z)=\sec z$ converging in the ring $\pi/2<|z|<3\pi/2$

$$C_m = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{m+1}}$$

Note that there are two poles inside any contour going around the ring $\pi/2 < |z| < 3\pi/2$; therefore :

$$\Rightarrow\ C_{-2} = \frac{1}{2\pi i}\oint \frac{z\,dz}{\cos z} = \operatorname{Res}\left(\frac{z}{\cos z};\frac{\pi}{2}\right) + \operatorname{Res}\left(\frac{z}{\cos z};-\frac{\pi}{2}\right) = \frac{\pi/2}{-\sin(\pi/2)} + \frac{-\pi/2}{-\sin(-\pi/2)} = \boxed{-\pi}$$

$$\Rightarrow\ C_{-4} = \frac{1}{2\pi i}\oint \frac{z^3\,dz}{\cos z} = \operatorname{Res}\left(\frac{z^3}{\cos z};\frac{\pi}{2}\right) + \operatorname{Res}\left(\frac{z^3}{\cos z};-\frac{\pi}{2}\right) = \frac{(\pi/2)^3}{-\sin(\pi/2)} + \frac{(-\pi/2)^3}{-\sin(-\pi/2)} = \boxed{-\frac{\pi^3}{4}}$$

---

6) Show that transformation $w = \frac{1}{2}\left(\frac{z}{e^a} + \frac{e^a}{z}\right)$, where α is a real constant, maps the interior of the unit circle into the exterior of the ellipse $\left(\frac{u}{A}\right)^2 + \left(\frac{v}{B}\right)^2 = 1$

Consider the mapping of the unit circle, $z=\exp(i\theta)$:

$$w = \frac{1}{2}\left(\frac{z}{e^a} + \frac{e^a}{z}\right) = \frac{1}{2}\left(\frac{e^{i\theta}}{e^a} + \frac{e^a}{e^{i\theta}}\right) = \frac{1}{2}\left(e^{i\theta-\alpha} + e^{-i\theta+\alpha}\right) = \cos\theta\,\frac{e^{-\alpha}+e^{\alpha}}{2} + i\,\sin\theta\,\frac{e^{-\alpha}-e^{\alpha}}{2}$$

$$= \underbrace{\cos\theta\cosh\alpha}_{u} - i\underbrace{\sin\theta\sinh\alpha}_{-v}$$

From the basic trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we obtain the result $\boxed{\left(\frac{u}{\cosh\alpha}\right)^2 + \left(\frac{v}{\sinh\alpha}\right)^2 = 1}$

To see that the interior of the unit circle is mapped to the exterior of this ellipse, consider the mapping of $z=0$: because the map has a pole at $z=0$, it is mapped to infinity, which is exterior to the ellipse in the w-plane

---

7) Find and sketch the domain of uniform convergence of series $F(z)=\sum_{n=1}^{\infty}\pi^{-n}\sin nz$ (use exponential representation of sine).

$$F(z) = \sum_{n=1}^{\infty}\pi^{-n}\sin nz = \sum_{n=1}^{\infty}\frac{e^{inz}-e^{-inz}}{2i\,\pi^n} = -\frac{i}{2}\sum_{n=1}^{\infty}\left(\frac{e^{iz}}{\pi}\right)^n + \frac{i}{2}\sum_{n=1}^{\infty}\left(\frac{e^{-iz}}{\pi}\right)^n$$

The two geometric series converge if the two geometric ratios are less than 1 in modulus:

$$\left.\begin{aligned} \left|\frac{e^{iz}}{\pi}\right| = \left|\frac{e^{i(x+iy)}}{2}\right| = \frac{e^{-y}}{\pi} < 1 \quad &\Rightarrow\quad e^{-y} < \pi \quad\Rightarrow\quad y > -\ln\pi \\ \left|\frac{e^{-iz}}{\pi}\right| = \left|\frac{e^{-i(x+iy)}}{2}\right| = \frac{e^{y}}{\pi} < 1 \quad &\Rightarrow\quad e^{y} < \pi \quad\Rightarrow\quad y < \ln\pi \end{aligned}\right\}$$

Converges within an infinite horizontal strip $\boxed{|y| < \ln\pi}$